应力应变分析

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内容提要

- 张量运算的基本法则
- 应力分析
- 应变分析

统计课程学习背景

- 弹塑性力学
  - 材料行为分析
- 有限元（弹性力学及有限元基础）
  - 单元、求解
- 程序结构力学（结构矩阵分析）
  - 编程

向量的表示方法

字母表示法

- 箭头

  $\vec{a}\ \vec{b}$

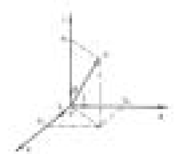
- 黑体字

  $\mathbf{a},\mathbf{b}$

坐标表示法

$$\mathbf{a} = a_x i + a_y j + a_z k$$

$$|\mathbf{a}| = \sqrt{a_x^{\ 2} + a_y^{\ 2} + a_z^{\ 2}}$$



矩阵表示法

$$\mathbf{a} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}$$

向量的数乘

$$\mathbf{b} = k\mathbf{a}$$

向量的和与差

$$\mathbf{a} = a_x i + a_y j + a_z k$$

$$\mathbf{b} = b_x i + b_y j + b_z k$$

$$\mathbf{a} + \mathbf{b} = (a_x + b_x)i + (a_y + b_y)j + (a_z + b_z)k$$

$$\mathbf{a} - \mathbf{b} = (a_x - b_x)i + (a_y - b_y)j + (a_z - b_z)k$$

向量的点积

$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$$

$$\mathbf{a}\cdot\mathbf{b}=\{\mathbf{a}\}^T\{\mathbf{b}\}$$

向量的叉积

$$|\mathbf{c}|=|\mathbf{a}||\mathbf{b}|\sin\theta$$

$$\mathbf{c}=\mathbf{a}\times\mathbf{b}$$

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

向量的夹角

$$\cos(\mathbf{a},\mathbf{b})=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\sin(\mathbf{a},\mathbf{b})=\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$$

3个向量的混合积

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

字母标记法

$$x, y, z$$

↓

$$x_1, x_2, x_3$$

↓

$$x_i$$

$$\sigma_{ij}=\sigma_{11},\sigma_{12},\sigma_{13},\sigma_{21},\sigma_{22},\sigma_{23},\sigma_{31},\sigma_{32},\sigma_{33}$$

求和约定

$$S=a_1x_1+a_2x_2+a_3x_3=\sum_{i=1}^{3}a_ix_i=a_ix_i$$

$$J_2=\frac{1}{2}S_{ij}S_{ij}$$

$$=\frac{1}{2}(S_{11}{}^2+S_{22}{}^2+S_{33}{}^2+S_{12}{}^2 \ldots S_{23}{}^2)$$

某一项的一个标号重复时，就表示将标号轮换取**1，2，3**时所得的各向之和

$$\varphi_{,i}dx_i=\frac{\partial\varphi}{\partial x_1}dx_1+\frac{\partial\varphi}{\partial x_2}dx_2+\frac{\partial\varphi}{\partial x_3}dx_3$$

注意

- 哑标变换字母标号时并不改变其含义

$$a_ib_i=a_jb_j=a_mb_m$$

$$a_{ij}x_j=a_{in}x_n$$

- 有括号的运算要特别注意

$$a_{ii}^2=a_{11}^2+a_{22}^2+a_{33}^2$$

$$(a_{ii})^2=(a_{11}+a_{22}+a_{33})^2$$

自由标号

- 不重复出现的标号称为自由标号

$$a_j=b_{ji}x_i=\begin{cases} a_1=b_{11}x_1+b_{12}x_2+b_{13}x_3 \\ a_2=b_{21}x_1+b_{22}x_2+b_{23}x_3 \\ a_3=b_{31}x_1+b_{32}x_2+b_{33}x_3 \end{cases}$$

Kronneker Delta δ_{ij}

$$\delta_{ij}=\begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases}$$

$$[\delta_{ij}]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

相关运算

$$\delta_{ij}\delta_{ij}=\delta_{ii}=\delta_{jj}=3$$

$$\delta_{ij}\delta_{jk}=\delta_{ik}$$

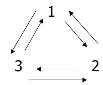
$$a_{ij}\delta_{ij}=a_{ii}=a_{jj}$$

$$a_j\delta_{ij}=a_i$$

$$\mathbf{e}_i\cdot\mathbf{e}_j=\delta_{ij}$$

置换符号 e_{ijk}

$$e_{ijk}=\begin{cases}1 & \text{如果下标为顺循环}\\ -1 & \text{如果下标为逆循环}\\ 0 & \text{如果下标重复}\end{cases}$$



$$e_{123}=e_{231}=e_{312}=1$$

$$e_{213}=e_{321}=e_{132}=-1$$

$$e_{113}=e_{322}=e_{122}=0$$

相关运算

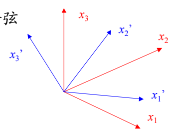
$$\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=e_{rst}a_{r1}a_{s2}a_{t3}$$

$$\mathbf{e}_i\times\mathbf{e}_j=\begin{cases}\mathbf{e}_k & \text{当}ijk\text{为顺循环}\\ -\mathbf{e}_k & \text{当}ijk\text{为逆循环}\\ 0 & \text{当}ijk\text{为非循环}\end{cases}$$

$$\mathbf{e}_i\times\mathbf{e}_j=e_{ijk}\mathbf{e}_k$$

空间坐标转换

- 假设在空间中存在一个坐标系 e_i
- 同时存在另一坐标系 e_i'
- 那么 $e_1'=l_{11}e_1+l_{12}e_2+l_{13}e_3=l_{1i}e_i$
- l_{1i} 为 e_i' 在 e_i 中的方向余弦



向量，一阶张量

$$\mathbf{u}=u_ie_i=u_i'e_i'$$

$$u_j=u_i'l_{ij}$$

$$\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}=\begin{bmatrix}l_{11} & l_{12} & l_{13}\\ l_{21} & l_{22} & l_{23}\\ l_{31} & l_{32} & l_{33}\end{bmatrix}\begin{Bmatrix}u_1'\\ u_2'\\ u_3'\end{Bmatrix}$$

定义：向量由3个分量确定，在坐标轴转动时，其分量服从坐标转轴公式

二阶张量

坐标系中 x_i 有一个量具有9个分量 a_{ij}，坐标转动后得到新的坐标系 x_i'，该量的9个分量变为 a_{ij}'

$$a_{ij}=a_{mn}'l_{mi}l_{nj}$$

张量相等

- 如果张量中的各个分量一一相等，则

$$a_{ij}=b_{ij}$$

张量加减与数乘

- 张量的加减为各个分量逐个加减运算
- 张量的数乘为张量的各个分量分别乘以系数

张量的并乘与张量的外积

$$\mathbf{ab}=\mathbf{a_ib_j}=\begin{Bmatrix}a_1\\ a_2\\ a_3\end{Bmatrix}\begin{Bmatrix}b_1 & b_2 & b_3\end{Bmatrix}=\begin{bmatrix}a_1b_1 & a_1b_2 & a_1b_3\\ a_2b_1 & a_2b_2 & a_2b_3\\ a_3b_1 & a_3b_2 & a_3b_3\end{bmatrix}$$

$$\mathbf{a_{ij}b_k}=\mathbf{d_{ijk}}$$

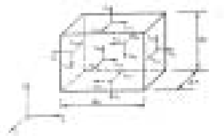
张量的缩并与张量的点积

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = c$$

$$a_{ik} b_{kj} = c_{ij}$$

$$c = \mathbf{a} : \mathbf{b} = a_{ij} b_{ij}$$

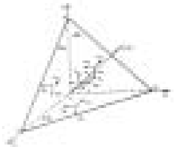
一点应力的表示方法



$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \quad [\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

任意斜截面上的应力



$$\sigma'_{mn} = \sigma_{ij} l_{im} l_{jn}$$

$$\begin{bmatrix} \sigma_{11}' & \sigma_{12}' & \sigma_{13}' \\ \sigma_{21}' & \sigma_{22}' & \sigma_{23}' \\ \sigma_{31}' & \sigma_{32}' & \sigma_{33}' \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

主应力

- 在某个坐标下，剪应力为零

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \sigma \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$$

应力状态不变量

$$\begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{vmatrix} = 0$$

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$$

$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2$$

应力偏量

$$\sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} = \frac{I_1}{3}$$

$$S_{ij} = \sigma_{ij} - \sigma_m \delta_{ij} = \begin{bmatrix} \sigma_{11}-\sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma_m & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_m \end{bmatrix}$$

应力偏量不变量

$$\left| S_{ij} - S_m \delta_{ij} \right| = 0$$

$$S^3 - J_1 S^2 - J_2 S - J_3 = 0$$

$$J_1 = S_{11} + S_{22} + S_{33} = 0$$

$$J_2 = -S_{11}S_{22} - S_{22}S_{33} - S_{11}S_{33} + S_{12}^2 + S_{23}^2 + S_{31}^2$$

$$J_3 = S_{11}S_{22}S_{33} + 2S_{12}S_{23}S_{31} - S_{11}S_{23}^2 - S_{22}S_{31}^2 - S_{33}S_{12}^2$$

主应力的求解方法（1）

- 用三次方程求根公式

$$S^3 - J_1 S^2 - J_2 S - J_3 = 0$$

主应力的求解方法（2）

- 等代三角形法

$$S = r\cos\theta$$

$$\cos^3\theta - \frac{J_2}{r^2}\cos\theta - \frac{J_3}{r^3} = 0$$

$$J_2 = -S_{11}S_{22} - S_{22}S_{33} - S_{11}S_{33} + S_{12}^2 + S_{23}^2 + S_{31}^2$$

$$J_3 = S_{11}S_{22}S_{33} + 2S_{12}S_{23}S_{31} - S_{11}S_{23}^2 - S_{22}S_{31}^2 - S_{33}S_{12}^2$$

$$\begin{cases} \dfrac{J_2}{r^2} = \dfrac{3}{4} \\ \dfrac{J_3}{r^3} = \dfrac{\cos 3\theta}{4} \end{cases} \qquad \begin{cases} r = \sqrt{\dfrac{4J_2}{3}} \\ \cos 3\theta = \dfrac{4J_3}{r^3} \end{cases}$$

便于手算

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \end{Bmatrix} + \sigma_m = \frac{2\sqrt{J_2}}{\sqrt{3}} \begin{Bmatrix} \cos\theta \\ \cos(\theta - \frac{2}{3}\pi) \\ \cos(\theta + \frac{2}{3}\pi) \end{Bmatrix} + \frac{I_1}{3}$$

主应力的求解方法（3）

- 求应力矩阵特征值方法

$$\begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{bmatrix}\begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

通用的数学软件库里面都有计算特征值的函数，可以直接使用

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例如

- Fortran IMSL

DEVCSF(3, SIG, 3, EVAL, EVEC, 3)

↑ 双精度 ↑ ↑ ↑

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} \quad \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

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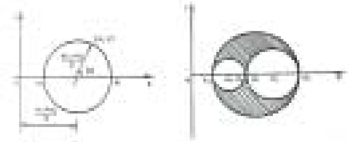
应力圆



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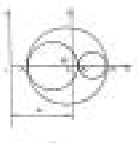
罗德参数

$$\mu_\sigma=\frac{MP_2}{MP_1}=\frac{2\sigma_2-\sigma_1-\sigma_3}{\sigma_1-\sigma_3}=\frac{2s_2-s_1-s_3}{s_1-s_3}$$

$\sigma_2=\sigma_3=0,\ \sigma_1>0,\ \mu_\sigma=-1$ 单拉

$\sigma_2=0,\ \sigma_1=-\sigma_3,\ \mu_\sigma=0$ 纯剪

$\sigma_2=\sigma_3=0,\ \sigma_1<0,\ \mu_\sigma=1$ 单压



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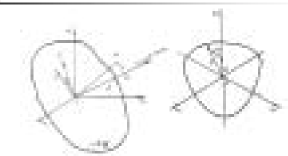
Haigh-Westergaard空间



$$\xi=\sqrt{3}\sigma_m \qquad \rho=\sqrt{2J_2} \qquad \cos 3\theta=\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}$$

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八面体正应力与剪应力

$$\sigma_{oct}=\frac{I_1}{3}=\sigma_m \qquad \tau_{oct}=\sqrt{\frac{2}{3}J_2}$$

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J_2的相关代表应力

等效应力

$$\bar{\sigma}=\sqrt{3J_2}=\frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$

等效剪应力

$$T=\sqrt{J_2}=\frac{1}{\sqrt{6}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$

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应力不变量之间的关系

参数	符号	关系
主应力	$\sigma_1,\ \sigma_2,\ \sigma_3$	
主应力偏量	$S_1,\ S_2,\ S_3$	$s_i=\sigma_i-\frac{I_1}{3}$
应力不变量	$I_1,\ I_2,\ I_3$	$I_1=\sigma_1+\sigma_2+\sigma_3$ $I_2=\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3$ $I_3=\sigma_1\sigma_2\sigma_3$
应力偏量不变量	$J_1,\ J_2,\ J_3$	$J_2=\frac{I_1^2}{3}-I_2$ $J_3=\frac{2I_1^3}{27}-\frac{I_1I_2}{3}+I_3$
几何参数	$\xi,\ \rho,\ \theta$	$\xi=\frac{I_1}{\sqrt{3}}$ $\rho=\sqrt{2J_2}$ $\theta=\frac{1}{3}\arccos\frac{3\sqrt{3}J_3}{2J_2^{3/2}}$
八面体应力	$\sigma_{oct},\ \tau_{oct}$	$\sigma_{oct}=\frac{I_1}{3}=\sigma_m$ $\tau_{oct}=\sqrt{\frac{2}{3}J_2}$
平均应力	$\sigma_m,\ \tau_m$	$\sigma_m=\frac{I_1}{3}$ $\tau_m=\sqrt{\frac{2}{5}J_2}$

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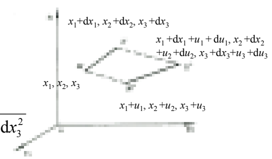
应变分析: 拉格朗日描述



$$dS=\sqrt{dx_1^2+dx_2^2+dx_3^2}$$

$$dS'=\sqrt{(dx_1+du_1)^2+(dx_2+du_2)^2+(dx_3+du_3)^2}$$

全微分

$$du_1=\frac{\partial u_1}{\partial x_1}dx_1+\frac{\partial u_1}{\partial x_2}dx_2+\frac{\partial u_1}{\partial x_3}dx_3$$

$$du_2=\frac{\partial u_2}{\partial x_1}dx_1+\frac{\partial u_2}{\partial x_2}dx_2+\frac{\partial u_2}{\partial x_3}dx_3$$

$$du_3=\frac{\partial u_3}{\partial x_1}dx_1+\frac{\partial u_3}{\partial x_2}dx_2+\frac{\partial u_3}{\partial x_3}dx_3$$

$$du_i=u_{i,j}x_j$$

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$$\begin{aligned}dS'^2-dS^2&=2\left\{\frac{\partial u_1}{\partial x_1}+\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_1}\right)^2+\left(\frac{\partial u_3}{\partial x_1}\right)^2\right]\right\}dx_1dx_1\\
&+2\left\{\frac{\partial u_2}{\partial x_2}+\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_2}\right)^2+\left(\frac{\partial u_2}{\partial x_2}\right)^2+\left(\frac{\partial u_3}{\partial x_2}\right)^2\right]\right\}dx_2dx_2\\
&+2\left\{\frac{\partial u_1}{\partial x_3}+\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_3}\right)^2+\left(\frac{\partial u_2}{\partial x_3}\right)^2+\left(\frac{\partial u_3}{\partial x_3}\right)^2\right]\right\}dx_3dx_3\\
&+\left\{\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right\}dx_1dx_2\\
&+\left\{\frac{\partial u_3}{\partial x_2}+\frac{\partial u_2}{\partial x_3}+\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}\right\}dx_2dx_3\\
&+\left\{\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_3}\right\}dx_1dx_3\end{aligned}$$

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拉格朗日应变

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}+u_{r,i}u_{r,j})$$

二阶项

$$\varepsilon_{11}=\frac{\partial u_1}{\partial x_1}+\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_1}\right)^2+\left(\frac{\partial u_3}{\partial x_1}\right)^2\right]$$

$$\varepsilon_{22}=\frac{\partial u_2}{\partial x_2}+\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_2}\right)^2+\left(\frac{\partial u_2}{\partial x_2}\right)^2+\left(\frac{\partial u_3}{\partial x_2}\right)^2\right]$$

$$\varepsilon_{33}=\frac{\partial u_1}{\partial x_3}+\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_3}\right)^2+\left(\frac{\partial u_2}{\partial x_3}\right)^2+\left(\frac{\partial u_3}{\partial x_3}\right)^2\right]$$

$$\varepsilon_{12}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right)$$

$$\varepsilon_{23}=\frac{1}{2}\left(\frac{\partial u_3}{\partial x_2}+\frac{\partial u_2}{\partial x_3}+\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}\right)$$

$$\varepsilon_{31}=\frac{1}{2}\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_3}\right)$$

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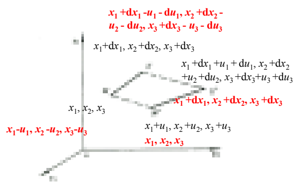
应变分析：欧拉描述



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$$\begin{aligned}dS'^2-dS^2&=2\left\{\frac{\partial u_1}{\partial x_1}-\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_1}\right)^2+\left(\frac{\partial u_3}{\partial x_1}\right)^2\right]\right\}dx_1dx_1\\
&+2\left\{\frac{\partial u_2}{\partial x_2}-\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_2}\right)^2+\left(\frac{\partial u_2}{\partial x_2}\right)^2+\left(\frac{\partial u_3}{\partial x_2}\right)^2\right]\right\}dx_2dx_2\\
&+2\left\{\frac{\partial u_1}{\partial x_3}-\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_3}\right)^2+\left(\frac{\partial u_2}{\partial x_3}\right)^2+\left(\frac{\partial u_3}{\partial x_3}\right)^2\right]\right\}dx_3dx_3\\
&+\left\{\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}-\left[\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right]\right\}dx_1dx_2\\
&+\left\{\frac{\partial u_3}{\partial x_2}+\frac{\partial u_2}{\partial x_3}-\left[\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}\right]\right\}dx_2dx_3\\
&+\left\{\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}-\left[\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_3}\right]\right\}dx_1dx_3\end{aligned}$$

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欧拉应变

$$E_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}-u_{r,i}u_{r,j})$$

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}+u_{r,i}u_{r,j})$$

$$E_{11}=\frac{\partial u_1}{\partial x_1}-\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_1}\right)^2+\left(\frac{\partial u_3}{\partial x_1}\right)^2\right]$$

$$E_{22}=\frac{\partial u_2}{\partial x_2}-\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_2}\right)^2+\left(\frac{\partial u_2}{\partial x_2}\right)^2+\left(\frac{\partial u_3}{\partial x_2}\right)^2\right]$$

$$E_{33}=\frac{\partial u_1}{\partial x_3}-\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_3}\right)^2+\left(\frac{\partial u_2}{\partial x_3}\right)^2+\left(\frac{\partial u_3}{\partial x_3}\right)^2\right]$$

$$E_{12}=\frac{1}{2}\left\{\frac{\partial u_1}{\partial x_2}-\left[\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right]\right\}$$

$$E_{23}=\frac{1}{2}\left\{\frac{\partial u_3}{\partial x_2}-\left[\frac{\partial u_2}{\partial x_3}+\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}\right]\right\}$$

$$E_{31}=\frac{1}{2}\left\{\frac{\partial u_3}{\partial x_1}-\left[\frac{\partial u_1}{\partial x_3}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_3}\right]\right\}$$

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柯西小应变张量

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$$

$$E_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}-u_{r,i}u_{r,j})$$

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}+u_{r,i}u_{r,j})$$

$$\varepsilon_{11}=\frac{\partial u_1}{\partial x_1}\qquad \varepsilon_{22}=\frac{\partial u_2}{\partial x_2}\qquad \varepsilon_{33}=\frac{\partial u_3}{\partial x_3}$$

$$\varepsilon_{12}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)\qquad \varepsilon_{23}=\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)\qquad \varepsilon_{31}=\frac{1}{2}\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right)$$

注意与工程应变的区别

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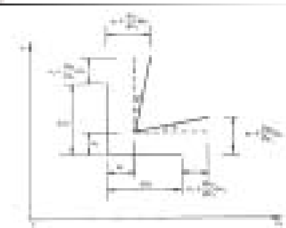
柯西应变的几何意义



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柯西应变与工程应变

$$\varepsilon=\{\varepsilon_{11}\quad \varepsilon_{22}\quad \varepsilon_{33}\quad \gamma_{12}\quad \gamma_{23}\quad \gamma_{31}\}^T$$

$$\gamma_{12}=\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}$$

$$[\varepsilon_{ij}]=\begin{bmatrix}\varepsilon_{11}&\varepsilon_{12}&\varepsilon_{13}\\ \varepsilon_{21}&\varepsilon_{22}&\varepsilon_{23}\\ \varepsilon_{31}&\varepsilon_{32}&\varepsilon_{33}\end{bmatrix}=\begin{bmatrix}\varepsilon_{11}&\frac{1}{2}\gamma_{12}&\frac{1}{2}\gamma_{13}\\ \frac{1}{2}\gamma_{21}&\varepsilon_{22}&\frac{1}{2}\gamma_{23}\\ \frac{1}{2}\gamma_{31}&\frac{1}{2}\gamma_{32}&\varepsilon_{33}\end{bmatrix}$$

其他概念

- 自学
  - 应变张量不变量
  - 应变偏量，应变偏量不变量
  - 八面体剪应变，八面体正应变
  - 等效应变
  - 等效剪应变

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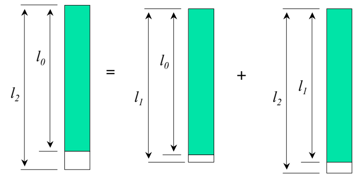
应变的可加性



$$\varepsilon = \frac{l_2 - l_0}{l_0} \neq \varepsilon_1 = \frac{l_1 - l_0}{l_0} + \varepsilon_2 = \frac{l_2 - l_1}{l_2}$$

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对数应变

$$d\varepsilon = \frac{dl}{l} \qquad \varepsilon = \int_{l_0}^{l} \frac{dl}{l} = \ln(\frac{l}{l_0})$$

$$\varepsilon_1 = \ln(\frac{l_1}{l_0})$$

$$\varepsilon_2 = \ln(\frac{l_2}{l_1})$$

$$\varepsilon = \ln(\frac{l_2}{l_0}) = \varepsilon_1 + \varepsilon_2$$

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作业

- 已知 $\sigma = \begin{bmatrix} 20 & 8 & 11 \\ 8 & 15 & 7 \\ 11 & 7 & 7 \end{bmatrix}$

  求主应力，主应力不变量，偏应力不变量